STABILITY OF AN INHOMOGENEOUS ELECTRON BEAM

I. A. Zhvaniya, R. Ya. Kucherov, and L. E. Rikenglaz

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 119–121, 1966

It has been shown in a series of theoretical papers devoted to the study of stationary plasma states that stationary periodic solutions of the self-consistent problem can exist [1–5].

There are considerable mathematical difficulties involved in examining the stability of these solutions in view of the inhomogeneity of the unperturbed state. The comparatively simple problem of the oscillations of charge density in electron beams with variable velocity was considered in [6, 7]. It was found that in a specific region of frequencies the wave may be amplified along the direction of motion of the beam.

A periodic plasma structure often arises in a bounded volume, and so it is of interest to determine whether the instability that arises is absolute or convective. It is with this in mind that the present paper solves the problem of the development of a perturbation in a nonuniform periodic electron beam.

We shall consider the one-dimensional problem of the passage of an electron beam with a constant particle flux density j through a uniform background of ions. If friction is neglected the stationary state of the system is described by the equations

$$j = nV,\quad {}^1/_2 mV^2 - e\varphi = {}^1/_2 mV^2_{\min} - e\varphi_{\min},$$
$$d^2\varphi \,/\, dx^2 = 4\pi e\,(n - N). \tag{1}$$

Here n, V, e, m are the density, velocity, charge and mass of the electrons, respectively, N is the density of ions, φ is the potential. Commencing the calculations at the point where the potential is a minimum and setting $\varphi_{\min} = 0$, we seek a solution of Poisson's equation which satisfies the conditions

$$\varphi(0) = 0, \qquad \varphi'(0) = 0. \tag{2}$$

It may easily be seen that in infinite space the solution will be periodic for all values $\alpha \equiv NV_{\min}/j$ except $\alpha = 1$.

Integrating Poisson's equation we obtain the following implicit form for the potential as a function of the coordinate:

$$\pm\frac{x}{\lambda\sqrt{2}} = \frac{1}{\alpha^{3/2}}\arccos\frac{\alpha\sqrt{1+\psi}-1}{\alpha-1} - {}$$
$$-\frac{\sqrt{2}}{\alpha}\left[\sqrt{1+\psi} - \frac{\alpha}{2}\psi - 1\right]^{1/2},$$
$$\frac{1}{\lambda^2} = \frac{4\pi e^2 J}{mV_{\min}^3},\qquad \varepsilon_0 = \frac{mV_{\min}^2}{2},\qquad \psi = \frac{e\varphi}{\varepsilon_0}. \tag{3}$$

Solving this equation for ψ in the limiting case of small amplitudes $(1-\alpha \ll 1)$, we obtain

$$\psi = 2(1-\alpha)\left(1 - \cos\frac{\alpha^{3/2}x}{\sqrt{2}\lambda}\right). \tag{4}$$

We shall consider the stability of this periodic solution as regards small longitudinal disturbances. Small oscillations of the electron beam are described by the following system of equations:

$$\frac{\partial V}{\partial t} + \frac{\partial}{\partial x}[V_0(x)V] = \frac{e}{m}\frac{\partial\varphi}{\partial x},$$
$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}[V_0(x)n] + \frac{\partial}{\partial x}[n_0(x)V] = 0,$$
$$\frac{\partial^2\varphi}{\partial x^2} = 4\pi en. \tag{5}$$

Here V, n, φ are the perturbations, and V_0, n_0, φ_0 are the unperturbed values of the velocity, density and potential.

Our initial condition is obtained by specifying the perturbations for t = 0

$$V(x,0) = V_1(x), \qquad n(x,0) = n_1(x). \tag{6}$$

We assume that the initial perturbations differ from zero in a finite region of space; then our boundary condition is given by the fact that there is no perturbation at infinity at the initial moment

$$V(\pm\infty, t) = n(\pm\infty, t) = \varphi(\pm\infty, t) = 0. \tag{7}$$

Multiplying equations (5), (7) by e^{-pt} and integrating from 0 to ∞ we obtain

$$pv + \frac{d}{dx}(V_0 v) - \frac{e}{m}\frac{d\Phi}{dx} = V_1(x),$$
$$p\rho + \frac{d}{dx}(V_0\rho) + \frac{d}{dx}(n_0 v) = n_1(x),\quad \frac{d^2\Phi}{dx^2} = 4\pi e\rho. \tag{8}$$

Here v, Φ, ρ represent the Laplace transforms of V, φ, n.

We eliminate Φ and ρ from system (8) and look for u (x, p) in the form

$$v(x,p) = \gamma(x,p)\,y(x,p),$$
$$\gamma(x,p) = \frac{1}{V_0(x)}\exp\left(-p\int_0^x\frac{dx'}{V_0(x')}\right). \tag{9}$$

As the result of transformations we obtain

$$y'' + f(x)y = \Gamma(x,p),$$
$$f(x) = \frac{4\pi e^2 n_0(x)}{mV_0^2(x)}, \tag{10}$$

$$\Gamma(x,p) = \frac{1}{V_0^2(x)\gamma(x,p)}\left(\frac{4\pi e^2}{m}\int_{-\infty}^x n_1(x')\,dx' + pV_1 + V_0\frac{dV_1}{dx}\right).$$

By $y_1(x)$, $y_2(x)$ we denote the fundamental solutions of the homogeneous equation (10). We then obtain for v (x, p)

$$v(x,p) = \frac{1}{WV_0(x)}\exp\left(-p\int_0^x\frac{dx'}{V_0(x')}\right)\int_{-\infty}^x\Gamma(x',p)\,K(x,x')\,dx'. \tag{11}$$

Here

$$W = y_1y_2' - y_2y_1' = \text{const},$$
$$K(x,x') = y_2(x)y_1(x') - y_1(x)y_2(x').$$

Carrying out the inverse Laplace transformation and changing the order of integration, we find

$$V(x,t) = \frac{1}{WV_0(x)}\left\{\int_{-\infty}^x\frac{K(x,x')}{V_0(x')}L(x')\,\delta\left(t - \int_{x'}^x\frac{dx''}{V_0(x'')}\right)dx' + {}\right.$$
$$\left. + \int_{-\infty}^x\frac{K(x,x')}{V_0(x')}V_1(x')\frac{d}{dt}\delta\left(t - \int_{x'}^x\frac{dx''}{V_0(x'')}\right)dx'\right\},$$
$$L(x') = \frac{4\pi e^2}{m}\int_{-\infty}^x n_1(x')\,dx' + V_0\frac{dV_1}{dx'}, \tag{12}$$

and after integrating over x'

$$V(x,t) = \frac{1}{WV_0(x)}\left\{K(x,x')L(x') + \frac{d}{dx}[K(x,x')V_1(x')]\right\},$$
$$\left(t = \int_{x'}^x\frac{dx''}{V_0(x'')}\right). \tag{13}$$

Since the initial disturbance differs from zero only in a finite region of space $(|x| < x_0)$, there exists, as is clear from formula (13), a value of time t_1 for any fixed value $x = x_1$ such that $|x'| > x_0$ for $t > t_1$ and the velocity V(x, t) vanishes. For another point $x = x_2$ $(x_2 > x_1)$ the corresponding moment of time $t_2 > t_1$. This means that the perturbation drift is the direction of increasing x. Using formula (13), we may easily ascertain that the drift velocity is equal to the velocity of the unperturbed beam.

The question as to whether the disturbance will increase or decrease in this case is determined by the behavior of $K(x, x')$ as a function of x'. The increase of $K(x, x')$, for example, indicates that the beam is convectively unstable. The solution obtained above loses its meaning if the wave breaks and multi-stream flow results. The condition necessary for the wave to break is $dx/dV = d^2x/dV^2 = 0$. It may easily be shown that (see formula (13)) these requirements are not fulfilled for all sufficiently smooth initial values of n_1 and V_1.

There are considerable mathematical difficulties involved in carrying out an analytic investigation of the properties of the fundamental solutions of equation (10). We thus confine ourselves to the case of small amplitudes when this equation reduces to Mathieu's equation

$$\frac{d^2y}{dz^2} + 4(1 - 3(\alpha - 1)\cos 2z)\, y = 0,$$

$$\left(z = \frac{x\alpha^{3/2}}{2\sqrt{2\lambda}},\ 1 - \alpha \ll 1\right). \tag{14}$$

It is a familiar fact [8] that this equation has one increasing solution. Using Whittaker's method [8], we seek the solution in the form $y = e^{\mu z}\Phi(z, \sigma)$, where μ and σ are new parameters, and Φ is a periodic function. We obtain as a result

$$y(z) \approx A_1 \exp\frac{\sqrt{7}\,q^2 z}{48}\sin 2z + A_2 \exp\left(-\frac{\sqrt{7}\,q^2 z}{48}\right)\cos 2z, \tag{15}$$

$$(q = 6(\alpha - 1)). \qquad (15)\ \text{(cont'd)}$$

It follows from formula (15) that the growth increment for small amplitudes is proportional to the square of the amplitude.

REFERENCES

1. I. B. Bernstein, J. M. Greene, and M. D. Kruskal, "Exact nonlinear plasma oscillations," Phys. Rev., vol. 108, p. 546, 1951.

2. P. L. Auer and N. J. Hurwitz, "Space charge neutralization by positive ions in diodes," J. Appl. Phys., vol. 30, p. 161, 1959.

3. S. D. Lebedev, Yu. Ya. Stavisskii, I. I. Bondarenko, and S.A. Maev, "Plasma oscillations when ion beams are neutralized," ZhTF, vol. 31, p. 1202, 1961.

4. Y. Ozawa, J. Kaji, and M. Kiton, "Nonlinear stationary plasma waves," J. Nucl. Energy C., vol. 6, p. 227, 1964.

5. R. Ya. Kucherov and L. E. Rikenglaz, "The periodic structure of a rarefied plasma," PMTF, no. 5, 1965.

6. P. V. Bliokh and Ya. Yu. Fainberg, "Charge-density waves in electron beams with variable velocity," ZhTF, vol. 26, p. 530, 1956.

7. O.E.H. Rydbeck, "Some wave propagation of the electron beam in vacuum and in ionized media," Nuovo Cimento Supplemento, vol. 10, p. 101, 1953.

8. N. W. MacLachlan, Theory and Application of Mathieu Functions [Russian translation], Moscow, NIL, 1953.

25 February 1965 Sukhumi